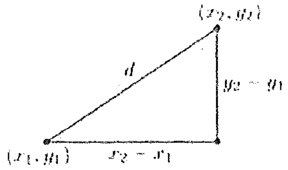
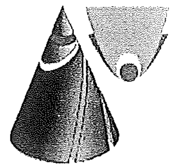
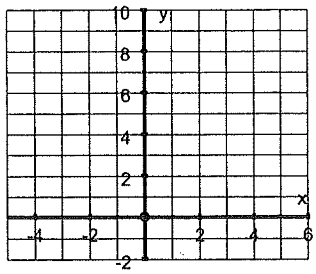


Analytic Geometry



Before we start looking at the conic sections in detail, we need to review some analytic geometry (also known as coordinate geometry). Most of this should be review for you.

1. Plot the points J(-4, 1) and K(5, 9), then find the distance between them.



Distance Formula: The distance *d* between two points (x_1, y_1) and (x_2, y_2) is given by

2. What theorem is this formula based on? Explain.

3. If the vertices of ΔXYZ are $X(-3,2)$, $Y(-1,-6)$, and $Z(5,0)$, prove that ΔXYZ is isosceles.

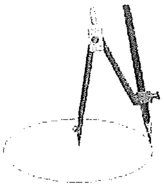
4. The three points $G(4,0)$, $H(h,6)$, and $I(7,1)$ are such that GH is twice the length of GI. Calculate the two possible values of h.

5. Go back to the diagram for Question 1. What are the coordinates of the midpoint of $\overline{JK}$?

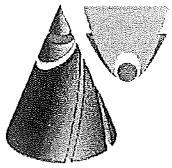
Midpoint Formula: The midpoint of the segment whose endpoints are (x_1, y_1) and (x_2, y_2) is:

6. Use coordinate geometry to prove the Triangle Midsegment Theorem:

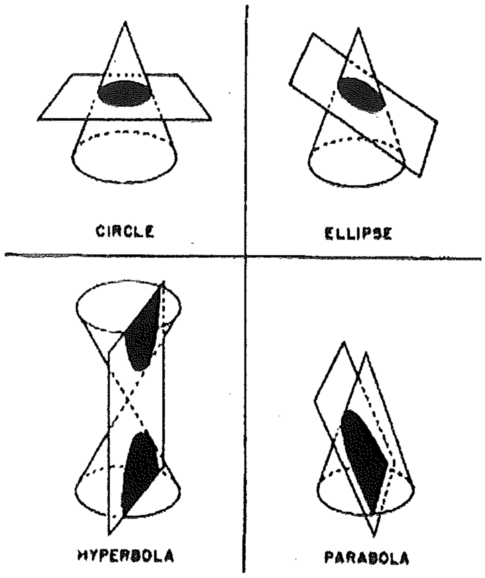
   The segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. [Hint: Let the vertices of the triangle be $X(0,0)$, $Y(a,b)$, and $Z(c,d)$.]



Circles



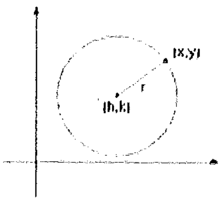
The main goal of this chapter is to develop an understanding for the geometric and algebraic properties of the *conic sections – circles, ellipses, hyperbolas* and *parabolas.* This figures are called the conic sections because they represent the 4 ways a plane can pass through a cone or double cone without passing through the vertex, as shown in the diagram.



We start with the circle.

Geometric definition of a circle: A circle with center *P* and radius *r* is the set of all points ...

If we graph a circle on the coordinate plane, and the coordinates of the center *P* are (h, k), then we get:



$$\sqrt{(x-h)^2+(y-k)^2}=r\text{, or}$$

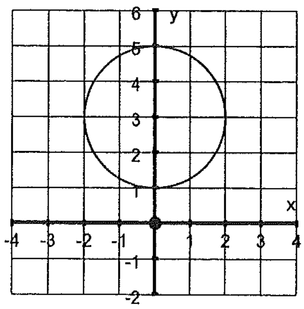
$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

which you may remember from Geometry as the **equation of a circle in standard form.**

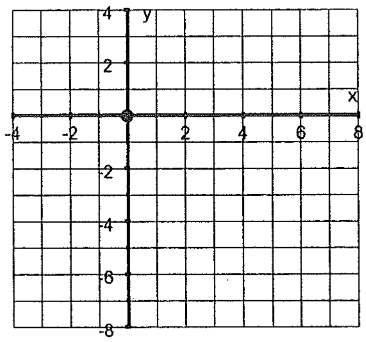
1. Write the equation of the circle in standard form.

a) center (-2, 4), radius = $\sqrt{10}$

b)



2. Graph the circle whose equation is $(x-1)^2 + (y+4)^2 = 9$.



Sometimes the equation is not written in standard form.

The general form of the equation of a circle is $x^2 + y^2 + Dx + Ey + F = 0$, where D, E, and F are constants.

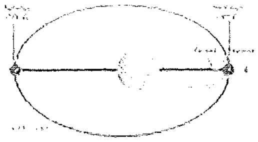
In this case, you need to put the equation in standard form before finding the center and radius or graphing. You can do this by completing the square (twice!)

3. The equation of a circle is $2x^2 + 2y^2 - 4x + 12y - 18 = 0$.

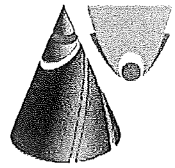
   a) Write the equation in standard form.

   b) Find the center and radius.

4. Write the standard form of the equation of a circle that passes through the points (5, 3), (-2, 2), and (-1, 5). Then identify the center and the radius.



Ellipses

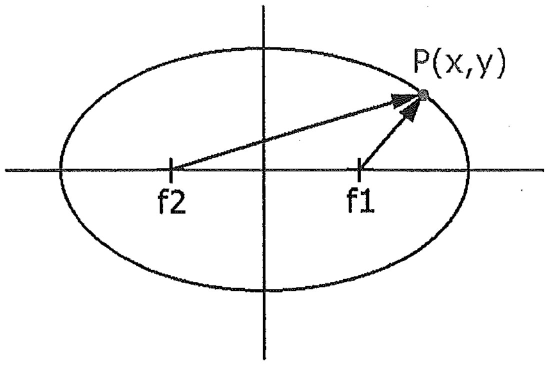


In the early 1600's Johannes Kepler demonstrated that planets orbit around the sun in elliptical orbits. Before this most astronomers attempted to model planetary orbits with increasingly complicated sets of circles on circles on circles, called epicycles.

Investigation

In this investigation, we will try to determine the geometric definition of an ellipse.

Every ellipse has two important points called *foci*. (Foci is the plural of focus.) The two foci of the ellipse shown are f1 and f2.



a) Measure the distance from f1 to P and from f2 to P to the nearest millimeter. Enter this information in the table below.

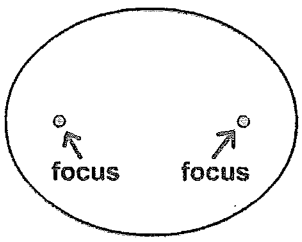
b) Choose two other points on the ellipse, and measure the distances from the foci.

	Distance from f1	Distance from f2	Sum of distances
Point P			
Point __			
Point __			

c) What do you observe?

Geometric definition of an ellipse: A circle with foci F_1 and F_1 is the set of all points ...

Besides the foci, there are some other important terms related to ellipses that you should know.



An ellipse has two axes. The longer one, which passes through the two foci, is called the ***major axis***. There is also a ***minor axis***. The two axes are perpendicular, and intersect at the ***center*** of the ellipse. The endpoints of the axes are called the ***vertices*** of the ellipse. The ***semi-major axis*** and the ***semi-minor axis*** are the segments from the center to the appropriate vertex.

Draw and label these terms on the ellipse shown.

Turning the geometric definition into an algebraic equation isn't that hard, but it is long, so we won't do it here.

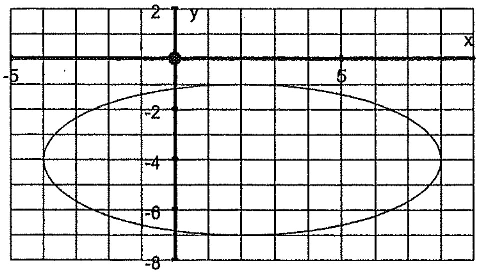
Standard form of the equation of an ellipse:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$ if the major axis is horizontal; and,

$$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$$ if the major axis is vertical.

In either case, the center of the ellipse is (h, k), a is the length of the semi-major axis, and b is the length of the semi-minor axis.

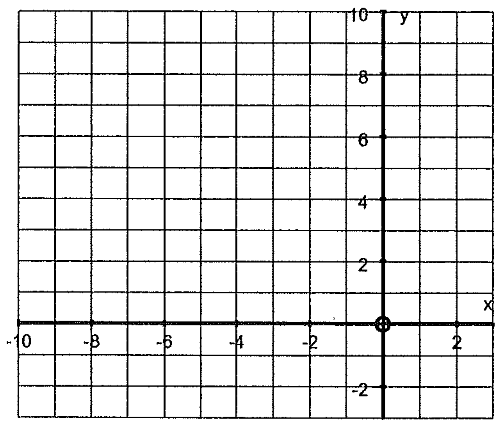
1. Write the equation in standard form of the ellipse shown.

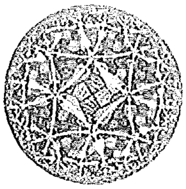


There is also a simple way to find the coordinates of the foci. The foci are located c units away from the center, going in either direction along the major axis, where $\boxed{c^2 = a^2 - b^2}$

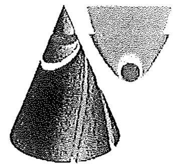
Find the coordinates of the foci of the above ellipse.

2. Given the ellipse $\frac{(y-3)^2}{25}+\frac{(x+4)^2}{9}=1$. Find the coordinates of the center, the foci, and the vertices. Then graph the equation.





Hyperbolas

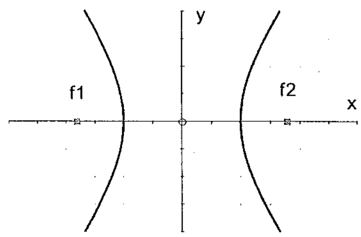


M. C. Escher's Circle Limit III, pictured above is based on Hyperbolic Geometry. In this type of non-Euclidean geometry, objects lie on a hyperbolic surface instead of a plane.

Investigation

Let's see if we can get this right this time! In this investigation, we will try to determine the geometric definition of a hyperbola.

Like an ellipse, a hyperbola has two *foci.*



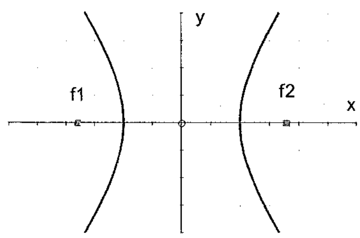
Choose three points on the hyperbola, and measure the distances from the foci.

	Distance from f1	Distance from f2	
Point ___			
Point ___			
Point ___			

What is true in each case?

Geometric definition of an hyperbola: A hyperbola with foci F_1 and F_1 is the set of all points ...

Vocabulary for Hyperbolas:



- Center
- Vertices – the point on each branch of the hyperbola that is closest to the center
- Asymptotes
- Transverse axis – the segment connecting the vertices – it's length is $2a$.
- Conjugate axis – perpendicular to the transverse axis with length $2b$ [Note that the conjugate axis is a segment not a line, even though it has no obvious endpoints.]

As with the ellipse, c is the distance from the center. However, in this case, $c > a$, and we the relationship between the parameters is $b^2 = c^2 - a^2$.

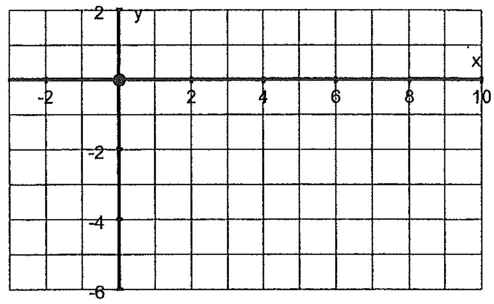
Indicate these features on the diagram.

3. The equation of an ellipse is given by $4x^2 + 9y^2 - 40x + 36y + 100 = 0$.

a) Write the equation in standard form.

b) Find the coordinates of the center, foci, and vertices.

c) Graph the equation.



One final property of an ellipse is its ***eccentricity***. The eccentricity is usually denoted by *e* and is defined as $e = \dfrac{c}{a}$.

4 a) What are the smallest and largest possible values for the eccentricity?

b) What does an ellipse look like if its eccentricity is small?

c) What does an ellipse look like if its eccentricity is large?

Standard form of the equation of an hyperbola:

If the transverse axis is horizontal: $$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

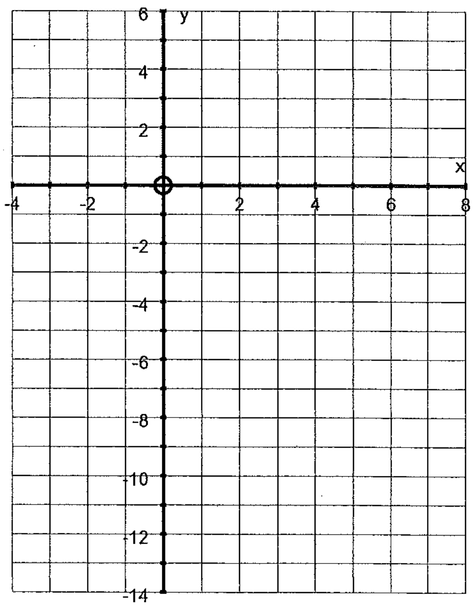
Asymptotes: $y-k=\pm\frac{b}{a}(x-h)$

If the transverse axis is vertical: $$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Asymptotes: $y-k=\pm\frac{a}{b}(x-h)$

In either case, the center of the hyperbola is (h, k), a is half the length of the transverse axis, and b is half the length of the conjugate axis.

1. Given the hyperbola $\frac{(y+4)^2}{36}-\frac{(x-2)^2}{25}=1$.



a) Find the coordinates of the center, the foci, and the vertices.

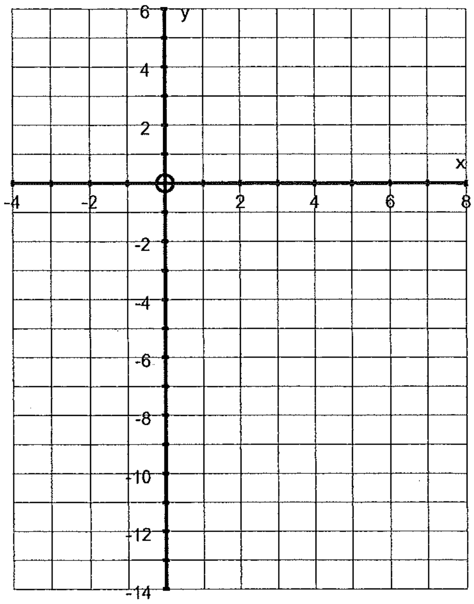
b) Find the equations of the asymptotes and graph.

To graph the asymptotes, you can draw a $2a \times 2b$ rectangle around the central area of the hyperbola, touching the vertices. Then the asymptotes go through the diagonals of the rectangle.

c) Graph the hyperbola.

2. Find the equation of the hyperbola with foci at (7, 1) and (-3, 1) and whose transverse axis is 8 units long.

3. Find the coordinates of the center, foci, and vertices, and the equations of the asymptotes of the graph of $9x^2 - 4y^2 - 54x - 40y - 55 = 0$. Then graph the hyperbola.



The *eccentricity* of a hyperbola is still defined as $e = \dfrac{c}{a}$.

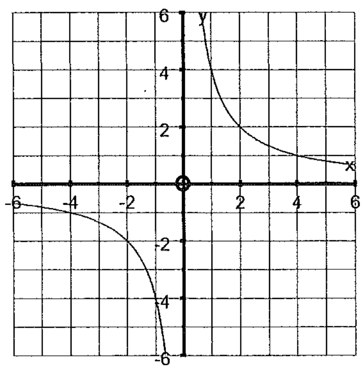
4. What is different about the eccentricity of a hyperbola as compared to an ellipse?

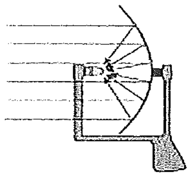
5. Do you remember another kind of hyperbola from Chapter 3?

   This is the graph of $y = \dfrac{4}{x}$.

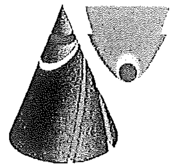
   Identify the asymptotes and the vertices.

   Draw the transverse axis on the graph.





Parabolas



Parabola-shaped mirrors and lenses are used in many applications from astronomical telescopes to solar camping water heaters because the parabolic shape focuses the reflected rays at a single point, the focus.

Of course, we have studied parabolas before, though we will focus on different properties this time. At the end of class, make sure you take a few minutes to consider what information is most easily found from $y = ax^2 + bx + c$ and what information is most easily found from the conic section-type equations we will look at today.

Notes:

- Geometric definition of a parabola:
- Standard form when directrix is vertical and axis of symmetry is horizontal:
- Standard form when directrix is horizontal and axis of symmetry is vertical:

Now for something different!

- Navigate to http://www.explorelearning.com, and click on "login."
- Username = "sas##" where ## is any two-digit number, password = "shanghai." (See who the computer thinks you are!)
- Click on "Browse Gizmos," then under College (!), click on "College Alg/Precalculus," then "Analytic Geometry."
- Under "Parabolas – Activity B," click on "Launch Gizmo." Make sure you use Activity B – otherwise you will be using a different form of the equation than is in our textbook.
- Click on the Exploration Guide link, and answer the questions.

Part 1 – The vertex and focus of a parabola

1.

2.

3 a)

b)

c)

4 a)

b)

c)

Part 2 – The geometric definition of the parabola

1 a)

b)

c)

2 a)

b)

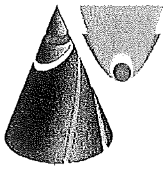
3 a)

b)

c)

d)

Try the assessment questions, then try Circles, Ellipses Activity A, or Hyperbolas Activity A.



Systems of Equations Based on Conics

The general equation for Conic Sections is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. In our work, the value of B is zero, except for the rectangular hyperbola $xy = k$. You should be able to identify the type of conic section just by looking at the equation.

1. Identify the type of conic section represented by each equation.

   a) $6x^2 + 3x - 4y - 12 = 0$

   b) $3y^2 + 2x^2 + 5y - x - 15 = 0$

   c) $4x^2 + 4y^2 + 5x - 2y - 63 = 0$

   d) $9x^2 - 27y^2 - 6x + 48y - 80 = 0$

2. Explain briefly how you can tell the type of conic section from the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Today we will be looking at what can happen if two conic sections are graphed on the same set of axes.

3. Complete the chart to show the appropriate number of intersections. Are all combinations possible? (Note: Circles have been left out of this chart. Why is this OK?)

	0 intersections	1 intersection	2 intersections	3 intersections	4 intersections
2 ellipses					
2 hyperbolas					
2 parabolas					

Ellipse/hyperbola					
Ellipse/parabola					
Hyperbola/parabola					

4. Solve each system of equations algebraically.

a) $$\begin{aligned} x - y &= 4 \\ x^2 &= 4y^2 + 13 \end{aligned}$$

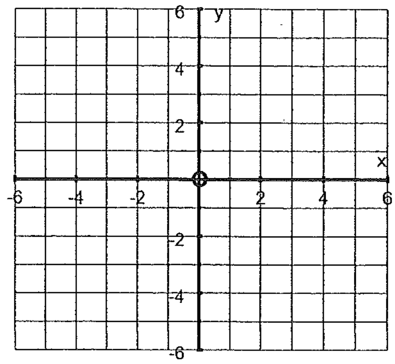
b) $$\begin{aligned} 9x^2 - 4y^2 &= 36 \\ x^2 + y^2 &= 4 \end{aligned}$$

c) $$\begin{aligned} 9x^2 + 25y^2 &= 225 \\ x^2 + y^2 - 2x &= 15 \end{aligned}$$

5. Solve graphically.

$x^2 + 4y^2 \leq 4$

$x^2 > y^2 + 1$



6. Show that the line $y = 2x + 2$ is tangent to the parabola $y^2 = 16x$ and find the coordinates of their point of contact.

7. Find the equations of the tangents drawn from $(-2,3)$ to $y^2 = 8x$.